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LOCAL EXPLICIT MANY-KNOT SPLINE HERMITE APPROXIMATION SCHEMES.(U)
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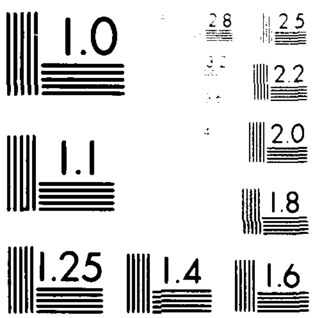

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LOCAL EXPLICIT MANY-KNOT SPLINE HERMITE APPROXIMATION SCHEMES

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ABSTRACT

If $f^{(i)}(\alpha)$ $(\alpha = a,b,\ i = 0,1,\dots,k-1)$ are given, then we get a class of the Hermite approximation operator $Qf = F$ satisfying $F^{(i)}(\alpha) = f^{(i)}(\alpha)$, where F is the many-knot spline function whose knots are at points $y_i : a = y_0 < y_1 < \dots < y_{k-1} = b$, and $F \in \mathbb{P}_k$ on $[y_{i-1}, y_i]$. The operator is of the form $Qf := \sum_{i=0}^{k-1} [f^{(i)}(a)\phi_i + f^{(i)}(b)\psi_i]$. We give an explicit representation of ϕ_i and ψ_i in terms of B-splines $N_{i,k}$. We show that Q reproduces appropriate classes of polynomials.

AMS (MOS) Subject Classification: 41A15

Key Words: Hermite interpolation, splines, local explicit, many-knot spline function.

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SIGNIFICANCE AND EXPLANATION

This paper deals with Hermite interpolation on the interval $[a,b]$ using many-knot splines. The contribution of this paper is to find a many-knot spline function F of degree $k - 1$ whose knots are at points y_i:

$$a = y_0 < y_1 < \dots < y_{k-1} = b .$$

When conditions are given on the ends of $[a,b]$: $f^{(i)}(\alpha)$, $\alpha = a,b$, $i = 0,\dots,k-1$, then $F = Qf = \sum_{i=0}^{k-1} [f^{(i)}(a)\phi_i + f^{(i)}(b)\psi_i]$ and $F \in \mathbf{P}_k$ on $[y_{i-1},y_i]$ $i = 1,2,\dots,k-1$. The explicit representations of basic functions ϕ_j, ψ_j which have properties

$$\phi_j^{(i)}(a) = \delta_{ij}, \quad \phi_j^{(i)}(b) = 0, \quad \psi_j^{(i)}(a) = 0, \quad \psi_j^{(i)}(b) = \delta_{ij}$$

$$\text{for all } \; i,j = 0,1,\dots,k-1$$

are given in terms of B-splines $N_{i,k}$. We also prove that this approximation operator Q reproduces appropriate classes of polynomials.

Since the degree of the many-knot splines used here is lower than that of ordinary Hermite interpolation and the knots of the splines can be chosen, it would be useful for some problems, for example, in Computer Aided Geometric Design (CAGD).

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LOCAL EXPLICIT MANY-KNOT SPLINE HERMITE APPROXIMATION SCHEMES

D. X. Qi* and S. Z. Zhou**

1. INTRODUCTION

Some authors considered operators of the form $Qf = \sum \lambda_i f N_{i,k}$, where $\{N_{i,k}\}$ is a sequence of B-splines and $\{\lambda_i\}$ is a sequence of linear functionals. The variation diminishing method of Schoenberg ([9], [5], [6]), the quasi-interpolant of de Boor and Fix are well-known. Such approximation schemes have several important advantages over spline interpolation. They can be constructed directly without matrix inversion, local error bounds can be obtained naturally. Qi considered so-called many-knot splines which have many more knots than degrees of freedom and constructed the cardinal spline $Qf = \sum f(x_i)q_{i,k}$, where $q_{i,k}$ is made up of B-splines on a uniform partition, has small support and satisfies $q_{i,k}(x_j) = \delta_{ij}$.[7] *Such an approximation operator reproduces* appropriate classes of polynomials[8].

The purpose of this paper is to construct a class of many-knot explicit local polynomial spline approximation operators for Hermite interpolation of real-valued functions defined on some interval $[a,b]$.

Let $\mathbb{P}_k$ be the set of polynomials of degree less than k, and let

$$a = y_0 < y_1 < \dots < y_{k-1} = b . \tag{1.0}$$

We define

$$\hat{S}_k := \{g : g|_{(y_i,y_{i+1})} \in \mathbb{P}_k,\ i = 0,1,\dots,k-2\} .$$

$\hat{S}_k$ is the familiar class of polynomial splines of order k with knots at the points y_i $(i = 0,1,\dots,k-2)$.

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Let F be a linear space of real valued functions on $[a,b]$, and suppose F contains the class of polynomials P_k. Given $f \in F$, we construct an approximation $F(\cdot) = Qf(\cdot)$ such that

$$F^{(\ell)}(a) = f^{(\ell)}(a), \quad F^{(\ell)}(b) = f^{(\ell)}(b) \qquad \ell = 0,1,\ldots,k-1. \tag{1.1}$$

In other words, set

$$Qf := \sum_{j=0}^{k-1} f^{(j)}(a)\phi_j(x) + \sum_{j=0}^{k-1} f^{(j)}(b)\psi_j(x) , \tag{1.2}$$

suppose ϕ_j, ψ_j satisfying

$$\phi_j^{(i)}(a) = \delta_{ij} , \qquad \phi_j^{(i)}(b) = 0 \tag{1.3}$$

$$\psi_j^{(i)}(a) = 0 , \qquad \psi_j^{(i)}(b) = \delta_{ij} \tag{1.4}$$

$$i,j = 0,1,\ldots,k-2 .$$

If ϕ_j and ψ_j are chosen in P_{2k-2}, then this problem above has been considered (see, for instance, [1], [3], [4]), and in this case $F \in P_{2k-2}$ on $[a,b]$.

We will find a many-knot spline $F \in \hat{S}_k$ satisfying (1.1). Such many-knot cardinal splines $\{\phi_j\}$ and $\{\psi_j\}$ are of degree less than k, therefore F is also of degree less than k. We present ϕ_j and ψ_j as explicit representations.

This paper proves that the many-knot spline Hermite approximation operator Q reproduces appropriate classes of polynomials on $[a,b]$.

2. CONSTRUCTION OF ϕ_j AND ψ_j

Without loss of generality, we assume $a = 0$ and $b = 1$. First of all set $k = 3$ as an example.

Let $\phi_0, \phi_1, \psi_0, \psi_1$ be piecewise polynomials of degree 2 with knots $x = \frac{1}{2}$, satisfying the following conditions

$$\phi_0(0) = 1 , \qquad \phi_1'(0) = 1 ,$$

$$\phi_0'(0) = \phi_0(1) = \phi_0'(1) = 0 , \qquad \phi_1(0) = \phi_1(1) = \phi_1'(1) = 0 ,$$

$$\phi_0\left(\tfrac{1}{2} + 0\right) = \phi_0\left(\tfrac{1}{2} - 0\right) , \qquad \phi_1\left(\tfrac{1}{2} + 0\right) = \phi_1\left(\tfrac{1}{2} - 0\right) ,$$

$$\phi_0'\left(\tfrac{1}{2} + 0\right) = \phi_0'\left(\tfrac{1}{2} - 0\right) , \qquad \phi_1'\left(\tfrac{1}{2} + 0\right) = \phi_1'\left(\tfrac{1}{2} - 0\right) .$$

and $\psi_0(x) := \phi_0(1 - x)$, $\psi_1(x) := -\phi_1(1 - x)$.

Easily one gets

$$\phi_0(x) = \begin{cases} -2x^2 + 1 , & x \in \left[0, \frac{1}{2}\right] , \\ 2(x - 1)^2 , & x \in \left[\frac{1}{2}, 1\right] ; \end{cases}$$

$$\phi_1(x) = \begin{cases} -\frac{3}{2} x^2 + x , & x \in \left[0, \frac{1}{2}\right] , \\ \frac{1}{2} (x - 1)^2 , & x \in \left[\frac{1}{2}, 1\right] . \end{cases}$$

Their graphs are sketched as follows

In order to consider the general case, denote

$$I_n := \{0,1,\dots,n\}$$

$$\phi_j(x) := \sum_{\mu \in I_{k-1}} \alpha_{j,\mu} x^u , \qquad x \in [x_i, x_{i+1}]$$

$$i \in I_{k-2} , \quad j \in I_{k-2}$$

(the partition is $0 = x_0 < x_1 < x_2 < \dots < x_{k-1} = 1$), and

$$\phi_j^{(\ell)}(x_i - 0) = \phi_j^{(\ell)}(x_i + 0), \quad i \in I_{k-2}\setminus\{0\}, \quad \ell \in I_{k-2} ,$$

$$\phi_j^{(i)}(0) = \delta_{ij} , \quad \phi_j^{(i)}(1) = 0 , \quad i,j \in I_{k-2} .$$

Since we have $k(k-1)$ unknown coefficients $\alpha_{j,\mu}$ with $k(k-1)$ conditions, so it seems possible to find $\alpha_{j,\mu}$. But, it is difficult to get the explicit representations for $\alpha_{j,\mu}$. Below we will directly present the explicit formulas for ϕ_j and ψ_j.

Here are the notations used in our discussion.

Let $\underline{X} := (x_i)$ be a nondecreasing sequence. The i-th B-spline of order k for the knot sequence (x_i) is denoted by

$$N_{i,k}(x) := (x_{i+k} - x_i)[x_i,\dots,x_{i+k}](\cdot - x)_+^{k-1}$$

for all $x \in \mathbb{R}$, where the symbol $[x_i,\dots,x_{i+k}]$ denotes the k-th oder divided-difference functional

$$\mathrm{sym}_\mu(\alpha_1,\alpha_2,\dots,\alpha_{n-1}) := \sum_{(\nu_1,\dots,\nu_\mu)} \alpha_{\nu_1}\alpha_{\nu_2} \cdots \alpha_{\nu_\mu}$$

$$\nu_j \in I_{n-1} , \quad \nu_i \neq \nu_j \ (i \neq j)$$

$$\xi_i^{(\mu)} := \mathrm{sym}_{\mu-1}(x_{i+1}, x_{i+1}, \dots, x_{i+k-1}) / \binom{k-1}{\mu-1}$$

$$\xi_i^{(0)} := \mathrm{sym}_0(\dots) := 1 .$$

From (1.0), we define

$$\begin{aligned} y_i - 1 &=: X_i , \\ y_i &=: X_{k-1+i} , \quad \text{for } i \in I_{k-1} . \end{aligned} \tag{2.2}$$

Thus we get a partition on $[-1,1]$ from $[0,1]$:

$$-1 = x_0 < x_1 < \dots < x_{k-2} < x_{k-1} = 0 < x_k < x_{k+1} < \dots < x_{2(k-1)} = 1 . \tag{2.3}$$

We construct the following functions on $[0,1]$ as a special kind of combination of B-splines

$$\phi_j(x) := \frac{1}{j!} \sum_{i \in I_{k-2}} \xi_i^{(j+1)} N_{i,k}(x) , \qquad \text{for } x \in [0,1] , \quad j \in I_{k-2} . \tag{2.4}$$

<u>Theorem 1.</u> The functions $\phi_j(x)$ defined in (2.4) satisfy

$$\phi_j^{(\ell)}(0) = \delta_{\ell j} , \tag{2.5}$$

$$\phi_j^{(\ell)}(x) = 0 \quad \text{for} \quad |x| > 1 , \qquad \ell, j \in I_{k-2} . \tag{2.6}$$

<u>Proof.</u> If $i \in I_{k-2}$ and $|x| > 1$, then $N_{i,k}^{(\ell)}(x) = 0$, therefore $\phi_j^{(\ell)}(x) = 0$ for all $\ell, j \in I_{k-2}$ and $|x| > 1$. If $i \bar{\in} I_{k-2}$, then $N_{i,k}^{(\ell)}(0) = 0$ since

$$I_{k-2} = \{i | i \in \{\dots,-2,-1,0,1,2,\dots\}, N_{i,k}(0) \neq 0\} .$$

By Marsden's Identity[6], for $x \in [0,1]$

$$x^{\mu-1} = \sum_{i \in I_{2k-3}} \xi_i^{(\mu)} N_{i,k}(x) , \qquad \mu = 1,2,\dots,k . \tag{2.7}$$

Thus

$$\begin{aligned} \phi_j^{(\ell)}(x) \Big|_{x=0} &= \Big(\frac{1}{j!} \sum_{i \in I_{k-2}} \xi_i^{(j+1)} N_{i,k}(x)\Big)^{(\ell)} \Big|_{x=0} \\ &= \frac{1}{j!} \Big[\Big(\sum_{i \in I_{k-2}} + \sum_{i=k-1}^{2k-3} \Big) \xi_i^{(j+1)} N_{i,k}(x)\Big]^{(\ell)} \Big|_{x=0} \\ &= \frac{1}{j!} (x^j)^{(\ell)} \Big|_{x=0} = \delta_{\ell j} , \quad \text{for} \quad \ell, j \in I_{k-2} . \end{aligned}$$

Let

$$\psi_j(x) := \phi_j(x-1)$$

Notice (2.2), (2.3), easily to see

$$\psi_j(x) = \frac{1}{j!} \sum_{i \in I_{k-2}} \xi_i^{(j+1)} N_{i+k-1,k}(x) \ . \tag{2.8}$$

By (2.5) we get

$$\psi_j^{(\ell)}(0) = 0 \ .$$

$$\psi_j^{(\ell)}(1) = \delta_{\ell j} \ , \quad \text{for} \quad \ell, j \in I_{k-2} \ .$$

Examples: $k = 3$,

$$\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha - \frac{1}{2}, & \alpha \end{pmatrix} \begin{pmatrix} N_{0,3}(x) \\ N_{1,3}(x) \end{pmatrix} ,$$

$$\alpha = sym_1(y_0,y_1)/2 = \frac{y_0 + y_1}{2} = \frac{y_1}{2} \ .$$

When the partition is uniform, then

$$\phi_0 = N_{0,3}(x) + N_{1,3}(x) \ ,$$

$$\phi_1 = -\frac{1}{4} N_{0,3}(x) + \frac{1}{4} N_{1,3}(x) \ .$$

$k = 4$,

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 - \frac{2}{3} & \alpha_1 - \frac{1}{3} & \alpha_1 \\ (\alpha_2 - \alpha_1 + \frac{1 - y_0}{3})/2! & (\alpha_2 - \alpha_1 + \frac{y_2}{3})/2! & \alpha_{2/2!} \end{pmatrix} \begin{pmatrix} N_{0,4}(x) \\ N_{1,4}(x) \\ N_{2,4}(x) \end{pmatrix}, \quad x \in [0,1]$$

where

$$\alpha_1 = sym_1(y_0,y_1,y_2)/3 = \frac{y_0 + y_1 + y_2}{3} ,$$

$$\alpha_2 = sym_2(y_0,y_1,y_2)/3 = \frac{y_0y_1 + y_1y_2 + y_2y_0}{3} \ .$$

In uniform case

$$\phi_0 = N_{0,4} + N_{1,4} + N_{2,4} ,$$

$$\phi_1 = -\frac{1}{3} N_{0,4} + \frac{1}{3} N_{2,4} , \qquad x \in [0,1]$$

$$\phi_2 = \frac{2}{54} N_{0,4} - \frac{1}{54} N_{1,4} + \frac{2}{54} N_{2,4} .$$

3. THE OPERATOR Q REPRODUCES APPROPRIATE CLASSES OF POLYNOMIALS

Using the functions ϕ_j and ψ_j, we have the following approximation operator

$$Qf(\cdot) := \sum_{j \in I_{k-2}} [f^{(j)}(0)\phi_j + f^{(j)}(1)\psi_j](\cdot) ,$$

Q defines a linear operator mapping F into $\hat{S}_k$.

Theorem 2. $Qg = g$ for all $g \in \mathbb{P}_k$.

Proof. Let

$$\text{span}(N) := \text{span}(N_{i,k};\ i \in I_{2k-3}) ,$$

$$\text{span}(\phi,\psi) := \text{span}(\phi_j,\psi_j;\ j \in I_{k-2}) ,$$

$$S := \{g : Qg = g\} .$$

Then both span(N) and span(ϕ,ψ) are linear subspaces of F on $[0,1]$ of dimension $2k - 2$.

By (2.4) and (2.8) we have

$$\text{span}(\phi,\psi) \subseteq \text{span}(N) .$$

Since

$$\dim(\text{span}(\phi,\psi)) = \dim(\text{span}(N)) = 2k - 2 ,$$

$$\text{span}(\phi,\psi) = \text{span}(N)$$

Obviously

$$\mathbb{P}_k \subseteq \text{span}(N)$$

i.e.

$$\mathbb{P}_k \subseteq \text{span}(\phi,\psi)$$

Now it is sufficient to prove that

$$S = \mathrm{span}(\phi,\psi) . \tag{3.1}$$

It follows from the definition of the set S and the operator Q that

$$S \subseteq \mathrm{span}(\phi,\psi) . \tag{3.2}$$

On the other hand, Theorem 1 implies that we have $Qf = f$ for any $f \in \mathrm{span}(\phi,\psi)$. Hence

$$\mathrm{span}(\phi,\psi) \subseteq S . \tag{3.3}$$

(3.2) and (3.3) mean that (3.1) is valid.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

If $f^{(i)}(\alpha)$ $(\alpha = a,b,\ i = 0,1,\ldots,k-1)$ are given, then we get a class of the Hermite approximation operator $Qf = F$ satisfying $F^{(i)}(\alpha) = f^{(i)}(\alpha)$, where F is the many-knot spline function whose knots are at points $y_i : a = y_0 < y_1 < \ldots < y_{k-1} = b$, and $F \in \mathbb{P}_k$ on $[y_{i-1}, y_i]$. The operator (cont.)

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ABSTRACT (cont.)

is of the form $Qf := \sum_{i=0}^{k-1} [f^{(i)}(a)\phi_i + f^{(i)}(b)\psi_i]$. We give an explicit representation of ϕ_i and ψ_i in terms of B-splines $N_{i,k}$. We show that Q reproduces appropriate classes of polynomials.

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